General Solutions of Several 1D Differential Equations

Yanting Wang

April 14, 2024

1. Time-Independent Schrödinger equation

In one dimension, the time-dependent Schrödinger equation of motion is

$$i\hbar\frac{\partial}{\partial t}\psi(x,t)=H\psi(x,t)\,, \tag{0.1}$$

where the Hamiltonian operator

$$H=-\frac{\hbar^2}{2m}\nabla^2+V\ . \tag{0.2}$$

Because variables x and t can be separated, its solution can be written as

$$\psi(x,t)=\psi(x)\exp\left(-i\frac{Et}{\hbar}\right) \tag{0.3}$$

where E is the energy value independent of either x or t. The wave function $\psi(x)$ satisfies the time-independent Schrödinger equation of motion

$$H\psi(x)=E\psi(x)\,, \tag{0.4}$$

which is

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x)-\frac{2m}{\hbar^2}V(x)\psi(x)+\frac{2m}{\hbar^2}E\psi(x)=0\ . \tag{0.5}$$

The way to solve the above equation depends on the specific form of $V(x)$.

2. Constant Potential

When the potential is a constant $V(x)=V_0$, Eq. (0.5) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x)+\frac{2m}{\hbar^2}(E-V_0)\psi(x)=0\,. \tag{0.6}$$

(1) If $E>V_0$, let $k=\sqrt{\frac{2m}{\hbar}(E-V_0)}$, Eq. (0.6) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x)+k^2\psi(x)=0\,, \qquad (0.7)\backslash$$

whose general solutions are

$$\psi(x)\sim\exp(\pm ikx)\,. \tag{0.8}$$

(2) If $E<V_0$, let $k=\sqrt{\frac{2m}{\hbar}(V_0-E)}$, Eq. (0.6) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x)-k^2\psi(x)=0\,, \tag{0.9}$$

whose general solutions are

$$\psi(x) \sim \exp(\pm kx) . \tag{0.10}$$

3. Potential Proportional to *x*

When $V(x)=k|x|$, give a total energy *E*, this potential has a classical turning point at $x=a$ when $E=ka$. we look at the region $x \geq 0$, Eq. (0.5) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x)-\frac{2mk}{\hbar^2}x\psi(x)+\frac{2m}{\hbar^2}E\psi(x)=0 . \tag{0.11}$$

Define $y \equiv \frac{x}{x_0}$ with $x_0=\left(\hbar^2/mk\right)^{1/3}$ and $\varepsilon \equiv \frac{E}{E_0}$ with $E_0=kx_0=\left(\hbar^2k^2/m\right)^{1/3}$, the above equation becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}\psi(y)-2(y-\varepsilon)\psi(y)=0 . \tag{0.12}$$

Further define $z \equiv 2^{1/3}(y-\varepsilon)$, we finally obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\psi(z)-z\psi(z)=0 , \tag{0.13}$$

which is called the Airy equation with the solution of the Airy function $\mathrm{Ai}(z)$, defined by

$$\mathrm{Ai}(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[i\left(zt+\frac{t^3}{3}\right)\right]\mathrm{d}t \tag{0.14}$$

4. Potential Proportional to *x*²

The simple harmonic oscillator has the potential

$$V(x)=\frac{1}{2}m\omega^2x^2 . \tag{0.15}$$

Eq. (0.5) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x)-\frac{m^2\omega^2}{\hbar^2}x^2\psi(x)+\frac{2m}{\hbar^2}E\psi(x)=0 . \tag{0.16}$$

Define $y \equiv \frac{x}{\sqrt{\hbar/m\omega}}$, $\varepsilon \equiv \frac{2E}{\hbar\omega}$, the above equation becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}\psi(y)+\left(\varepsilon-y^2\right)\psi(y)=0 , \tag{0.17}$$

whose solution can be written as

$$\psi(y)=h(y)\exp\left(-\frac{y^2}{2}\right) \tag{0.18}$$

with $h(y)$ satisfying

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}h(y)-2y\frac{\mathrm{d}}{\mathrm{d}y}h(y)+(\varepsilon-1)h(y)=0 . \tag{0.19}$$

The general solution of the above equation is the Hermite polynomials:

$$h(y)=H_n(y) , \tag{0.20}$$

where $n = 0, 1, 2, \cdots$ and $\varepsilon = 2n + 1$.

5. Normalization and Boundary Conditions

The general solutions must satisfy the normalization condition

$$\int \psi^*(x)\psi(x)\mathrm{d}x = 1, \tag{0.21}$$

along with the boundary conditions:

(1) $\psi(x) \to 0$ when $x \to \pm\infty$ (except the plane wave);

(2) At the joint point a, $\psi(a^-) = \psi(a^+)$ and $\psi'(a^-) = \psi'(a^+)$, which can be combined together to require $\frac{\mathrm{d}}{\mathrm{d}x}\ln\psi(a^-) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi(a^+)$.